# Advanced Systems Theory

## 03/11/2021, Wednesday, 08:30 – 11:30

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**1** $(6 + 12 = 18$ pts) **Disturbance decoupling with measurement feedback**

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Consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu + Ed \\ y &= Cx \\ z &= Hx \end{aligned}$$

with

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, H = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}.$$

**(a)** Show that the problem of disturbance decoupling (from $d$ to $z$) by measurement feedback is solvable.

**(b)** Compute a dynamic controller that makes the system from $d$ to $z$ decoupled.

**2** (18 pts) **Tracking and regulation**

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Consider the system given as the interconnection of the exosystem

$$\dot{x}_1(t) = 0$$

and the to-be-controlled system

$$\begin{bmatrix} \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1(t) + \begin{bmatrix} -1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

$$z(t) = -x_1(t) + x_2(t) - x_3(t) + x_4(t).$$

Construct a regulator.

**3** (18 pts) **Conditioned invariant subspaces**

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Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two $(C, A)$-invariant subspaces. Prove that there exists a matrix $G$ such that $(A + GC)\mathcal{S}_i \subseteq \mathcal{S}_i$ for $i = 1, 2$ if and only if $\mathcal{S}_1 + \mathcal{S}_2$ is $(C, A)$-invariant.

## 4 (6 + 12 = 18 pts) **Lyapunov equation**

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t). \tag{$\star$}$$

Assume that $K, M, N$ are matrices such that $M \geqslant 0$, $N = -N^T$, $K > 0$, $A^T K + KA = 0$, and $(MB^T K, A)$ is observable.

**(a)** Show that all eigenvalues of $A$ are on the imaginary axis.

**(b)** Show that the feedback $u = Fx$ where $F = (N - M)B^T K$ stabilizes the system $(\star)$.

## 5 (6 + 6 + 6 = 18 pts) **Synchorization**

Consider $p$ agents having the dynamics

$$\dot{x}_i = Ax_i + Bz_i$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

for $i \in \{1, 2, \ldots, p\}$. Let $G = (V, E)$ be a undirected simple connected graph where $V = \{1, 2, \ldots, p\}$. We want to find $K$ such the diffusive coupling

$$z_i = K \sum_{\{i,j\} \in E} (x_j - x_i)$$

synchronizes the overall multi-agent system. To do so, we aim at solving the Riccati equation

$$A^T P + PA - PBB^T P + Q = 0$$

where

$$Q = \begin{bmatrix} 15 & 0 \\ 0 & 10 \end{bmatrix}.$$

**(a)** Write down the Hamiltonian matrix $H$ and show that its eigenvalues are $-2$ and $2$.

**(b)** Show that

$$(H^2 + 4H + 4I) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 16 & 3 \\ 3 & 4 \end{bmatrix} = 0.$$

**(c)** Find $P > 0$ satisfying the Riccati equation.

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10 pts free